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Geometric Dual and Matrix Theory for SO/Sp Gauge Theories

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Abstract: In this paper, we give a proof of the equivalence of $\mathcal{N}=1$ SO/Sp gauge theories deformed from $\mathcal{N}=2$ by the superpotential of adjoint field Φ, the dual type IIB superstring theory on CY threefold geometries with fluxes and orientifold action after geometric transition. Furthermore, by relating the geometric picture to the matrix model, we show the equivalence between the field theory and the corresponding matrix model.

Keywords: Geometric Dual, Matrix Model, SO and Sp Gauge Theory.

Contents

1. Introduction

A few months ago, a deep relationship between matrix models and supersymmetric gauge field theories has been pointed out in [1, 2, 3]. In these papers, it was shown that exact glueball effective actions for supersymmetric gauge field theories can be calculated by planar diagrams of corresponding matrix models. Since then, a lot of works have been done to check this conjecture by explicit examples (e.g. [4] to [16], and [40] to [60]), from which some remarkable features of the new method have been demonstrated. For example, in [4, 5, 6, 10] it was shown that different massive vacua of the mass deformed $\mathcal{N}=4$ theory are related to each other by $SL(2,Z)$ modular groups, so the Montonen-Olive duality is not an assumption, but rather a derived result. It was also shown that the matrix model can calculate not only the exact low energy superpotential, but also quantum corrections of classical moduli spaces.

Among these results, two papers [15] and [46] used purely the language of field theories to prove the DV conjecture. These two proofs are very useful because they do not rely on the geometric picture and explain why calculations of effective actions can be reduced to matrix models. They provide also bases to generalize to other interesting cases, for example, the double trace deformation studied in [61] or the SO/Sp gauge groups studied in [58].

With these developments, it seems that to prove the DV conjecture, the geometric picture is not really needed. However, the field theory proof is not very general at this moment and for theories which can be geometrically engineered and embedded into string theory, the geometric method has been proven to be a very useful alternative. One explicit example can be found in [1] for the $\mathcal{N}=1$ $U(N)$ theory with one adjoint field Φ and arbitrary superpotential $W=\sum_{r=1}^{n+1} g_r u_r=\sum_{i=1}^{n+1} g_r \operatorname{Tr}(\Phi^r)/r$. The proof in [1] was based on works of [17, 18, 19, 20] where it was shown by large N duality that the calculation in the field theory is equivalent to the one in the dual geometry. So if we can derive the dual geometry (the spectral curve and periods of cycles) from the matrix model, by the link between the geometry and field theory, the relationship between the matrix model and field theory is established also.

In this paper, we will use the same logic to extend the proof of DV conjecture to $\mathcal{N}=1$ $SO(N)/Sp(2N)$ theories with one adjoint field Φ and arbitrary superpotential $W=\sum_{r=1}^{n+1} g_{2r} u_{2r}=\sum_{r=1}^{n+1} g_{2r} \operatorname{Tr}(\Phi^{2r})/2r$. We will show first that the exact effective action calculated by field theory method is same as the one calculated by the dual geometry method. Then we derive the corresponding spectral curve from the matrix model and match physical quantities such as S_i and Π_i at two sides of the matrix model and dual geometry. Combining the first step, it will complete our proof of DV conjecture for SO/Sp gauge groups.

The organization of the paper is following. In section two we provide the analysis in field theory. In section three we review the geometric dual picture and give a proof of the equivalence between the gauge theory and dual geometry. In section four, we present the derivation of the dual geometry from the matrix model, thus close the loop of our proof.[1]

2. The analysis in field theory

First let us analyze the classical moduli space of $SO(2N)$, $SO(2N+1)$ and $Sp(2N)$ (the notation for $Sp(2N)$ is that the rank of the gauge group is N) with following superpotential

$$W=\sum_{r=1}^{n+1} g_{2r}u_{2r}=\sum_{r=1}^{n+1} g_{2r}\frac{\mathrm{Tr}(\Phi^{2r})}{2r}\ . \tag{2.1}$$

By gauge transformations, we can rotate the Φ into following form: $diag(x_1 i\sigma_2,...,x_N i\sigma_2)$ for $SO(2N)$, $diag(x_1 i\sigma_2,...,x_N i\sigma_2,0)$ for $SO(2N+1)$ and $diag(x_1,-x_1,...,x_N,-x_N)$ for $Sp(2N)$ with σ_2 the Pauli matrix. The supersymmetric vacua are given by solutions of F-terms, i.e., roots of

$$W'(x)=g_{2n+2}x\prod_{j=1}^{n}(x^2\pm a_i^2),\qquad \pm\quad for\quad SO/Sp. \tag{2.2}$$

If we choose N_i x_i to be the same value $ia_i(a_i)$ (with the convention that $a_0=0$), the gauge group is broken to

$$SO(2N)\rightarrow SO(2N_0)\prod_{j=1}^{n}U(N_j) \tag{2.3}$$

with $\sum_{j=0}^{n}N_j=N$ for $SO(2N)$ (for $SO(2N+1)$ or $Sp(2N)$, $SO(2N_0)$ is replaced by $SO(2N_0+1)$ or $Sp(2N_0)$). At low energy, $SO(2N_0)$, $SO(2N_0+1)$, $Sp(2N_0)$ as well as $SU(N_i)$ develop a mass gap and confine, so there are n massless $U(1)$ gauge fields left. It is the exact effective action for these fields we are looking for.

Since our theories are deformed from corresponding $\mathcal{N}=2$ theories by the superpotential (2.1), we can calculate the exact superpotentials for these deformed theories by using the well-known Seiberg-Witten curves [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. The method has been elaborated in [17, 32].

[1]When submitting this paper, we noticed that two papers [62] and [63] have some overlaps with this paper.

The basic idea is that $\mathcal{N}=2$ theories deformed only by W of (2.1) have unbroken supersymmetries on a submanifold of Coulomb branches, where there are additional l massless fields besides u_r, such as magnetic monopoles or dyons. The exact low energy superpotential in these vacua is given by

$$W_{eff}=\sum_{r=1}^{n+1} g_{2r}\langle u_{2r}\rangle \tag{2.4}$$

with $\langle u_{2r}\rangle$ taking value in the submanifold where l monopoles are massless. In other words, we require l mutually local monopoles or dyons in the submanifold of Coulomb branches. This requirement put l conditions in original Coulomb branches and $\langle u_{2r}\rangle$ lie on the codimension l submanifold.

Because $\langle u_{2r}\rangle$ lie on the codimension l submanifold, we can parameterize them by $(N-l)$ parameters. To get the low energy effective action, we need to minimize W_{eff} in (2.4) regarding these parameters and substitute results back to W_{eff}. By this way, we get the low energy effective action W_{low} as a function of g_{2r} and Λ only.

The above conditions can be translated into the requirement of proper factorization forms of corresponding Seiberg-Witten curves as shown in [17, 32]. For SO/Sp gauge groups, as remarked in [32], there are two forms of SW curves. One is as a hyperelliptic curve of genus N in [30] and another is as a hyperelliptic curve of genus $2N$ with Z_2 symmetry in [28, 27, 31]. It was found that to connect to the geometric picture, the second choice is more natural and will be used throughout the paper.

Let us see how it works by the example of $SO(2N)$. $SO(2N)$ can be embedded into $U(2N)$ and considered as the Z_2 quotient of later. With the superpotential (2.1) (2.2), $U(2N)$ is broken to $2n+1$ factors as

$$U(2N)\rightarrow U(2N_0)\prod_{j=1}^{n} U(N_{j+})\times U(N_{j-})$$

with $2N_0+\sum_{j=1}^{n}(N_{j+}+N_{j-})=2N$ and the corresponding SW curve is factorized as

$$y^2=F_{2(2n+1)}(x)H_{2N-(2n+1)}(x)^2\ .$$

However, to reduce to $SO(2N)$ group, we must take the Z_2 action which requires $N_{j+}=N_{j-}$. The Z_2 action also maps $U(N_{j+})$ located at ia_j to $U(N_{j-})$ located at $-ia_j$ and projects $U(2N_0)$ located at 0 to $SO(2N_0)$, so finally we get the breaking pattern

$$SO(2N)\rightarrow SO(2N_0)\prod_{j=1}^{n} U(N_j).$$

Considering the Z_2 action of the factorized SW curve, we get [32]

$$y^2=P_{2N}(x^2,u)^2-4\Lambda^{4N-4}x^4=(xH_{2N-(2n+2)}(x))^2F_{2(2n+1)}(x),$$

where both $H(x)$ and $F(x)$ are functions of x^2. Knowing the factorized form, the gauge coupling constants of the remaining massless $U(1)$ fields can be calculated by the period matrix of the reduced curve

$$y^2 = F_{2(2n+1)}(x^2; \langle u_{2r}\rangle) = F_{2(2n+1)}(x^2; g_{2r}, \Lambda) \tag{2.5}$$

As we will show shortly, the function $F_{2(2n+1)}(x^2)$ is related to the deformed superpotential and geometry by

$$g_{2n+2}^2 F_{2(2n+1)}(x^2) = W'(x)^2 + f_{2n}(x)$$

where $f_{2n}(x)$ with degree $2n$ is a function of x^2.

2.1. Rephrasing the problem

As shown in [17, 20], the factorization and extremum can be restated into a pure algebraic problem which is well posed and has a unique solution: *Find $P_{2N}(x;u)$ such that*[2]

$$\begin{aligned}
SO(2N):\quad & P_{2N}(x^2,u)^2 - 4\Lambda^{4N-4}x^4 = x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x) \\
& = x^2 H_{2N-2n-2}^2(x)\frac{1}{g_{2n+2}^2}(W'(x)^2 + f_{2n}(x)), \qquad (2.6)\\
SO(2N+1):\quad & P_{2N}(x^2,u)^2 - 4\Lambda^{4N-2}x^2 = x^2 H_{2N-2n-2}^2(x) F_{2(2n+1)}(x) \\
& = x^2 H_{2N-2n-2}^2(x)\frac{1}{g_{2n+2}^2}(W'(x)^2 + f_{2n}(x)), \qquad (2.7)\\
Sp(2N):\quad & [x^2 P_{2N}(x^2,u) + 2\Lambda^{2N+2}]^2 - 4\Lambda^{4N+4} = x^2 H_{2N-2n}^2(x) F_{2(2n+1)}(x) \\
& = x^2 H_{2N-2n}^2(x)\frac{1}{g_{2n+2}^2}(W'(x)^2 + f_{2n}(x)), \qquad (2.8)
\end{aligned}$$

where $W'(x) = g_{2n+2}x\prod_{i=1}^n(x^2 \pm a_i^2)$ (where $+$ for SO and $-$ for Sp) is given, together with following boundary conditions at $\Lambda \to 0$ as

$$SO(2N):\quad P_{2N}(x^2,u) \to x^{2N_0}\prod_{i=1}^n (x^2+a_i^2)^{N_i}, \qquad \sum_{i=0}^n N_i = N, \tag{2.9}$$

$$SO(2N+1):\quad P_{2N}(x^2,u) \to x^{2N_0}\prod_{i=1}^n (x^2+a_i^2)^{N_i}, \qquad \sum_{i=0}^n N_i = N, \tag{2.10}$$

$$Sp(2N):\quad P_{2N}(x^2,u) \to x^{2N_0}\prod_{i=1}^n (x^2-a_i^2)^{N_i}, \qquad \sum_{i=0}^n N_i = N, \tag{2.11}$$

Above boundary conditions mean that gauge groups are broken as

$$SO(2N) \to SO(2N_0) \times \prod_{i=1}^n U(N_i),$$

[2]Following discussions are under the assumption that the wrapping number $N_i \neq 0$ for all $i = 0, ..., n$ which is also used in [20]. The discussion for more general cases is under investigation.

$$SO(2N+1) \to SO(2N_0+1) \times \prod_{i=1}^{n} U(N_i),$$
$$Sp(2N) \to Sp(2N_0) \times \prod_{i=1}^{n} U(N_i).$$

Using same method as in [20] it can be proved that solutions for above problems are unique.

Once the low energy effective action

$$W_{low}(g_{2r}, \Lambda) = \sum_{r=1}^{n+1} g_{2r} \langle u_{2r} \rangle$$

is obtained, we can calculate

$$\frac{\partial W_{eff}}{\partial g_{2r}} = \langle u_{2r} \rangle, \tag{2.12}$$
$$\frac{\partial W}{\partial \log(\Lambda^{2\hat{N}})} = \frac{-b_{2n}}{4 g_{2n+2}}. \tag{2.13}$$

where $\hat{N}$ is $2N-2$ for $SO(2N)$, $2N-1$ for $SO(2N+1)$ and $2N+2$ for $Sp(2N)$. The S_0 is the glueball superfield for $SO(2N_0)$, $SO(2N_0+1)$ or $Sp(2N_0)$ factor and S_i is the glueball superfield for $U(N_i)$ factor. The b_{2n} is the leading coefficient of the function $f_{2n}(x)$. We will give derivations of these results in next subsection.

2.2. The function $F_{2(2n+1)}(x)$

As we mentioned above, the function $F_{2(2n+1)}(x)$ is related to the deformed superpotential and geometry by

$$g_{2n+2}^2 F_{2(2n+1)}(x^2) = W'(x)^2 + f_{2n}(x) \ . \tag{2.14}$$

This result has been given in [32] for SO groups. Here we adopt the method in [17, 20] which will also enable us to show relation (2.13).

Let us start with the $SO(2N)$ gauge group. In this case, the SW curve is factorized as

$$P_{2N}(x^2,u)^2 - 4\Lambda^{4N-4}x^4 = (\widetilde{H}_{2N-2n-1}(x))^2 F_{2(2n+1)}(x) = (xH_{2N-(2n+2)}(x))^2 F_{2(2n+1)}(x)$$

Notice that since both the left hand side and $F_{2(2n+1)}(x)$ are functions of x^2 and the degree of $\widetilde{H}_{2N-2n-1}(x)$ is odd, one factor x must be factorized out in $\widetilde{H}_{2N-2n-1}(x)$, thus we can write $\widetilde{H}_{2N-2n-1}(x) = xH_{2N-(2n+2)}(x)$. For our convenience, we change it to

$$(\frac{P_{2N}(x^2,u)}{x^2})^2 - 4\Lambda^{4N-4} = (H_l(x))^2 x^{-2} F_{4N-2l-2}(x), \tag{2.15}$$

with $\frac{P_{2N}(x^2,u)}{x^2}$ a polynomial of x^2. As in [20], the problem of factorizing the SW curve and minimizing the superpotential under these constraints can be translated into minimizing the following superpotential

$$W = \sum_{r=1}^{n+1} g_{2r}u_{2r} + \sum_{i=1}^{l}[L_i((\frac{P_{2N}(x^2,u)}{x^2})|_{x=p_i} - 2\epsilon_i\Lambda^{2N-2}) + Q_i\frac{\partial(\frac{P_{2N}(x^2,u)}{x^2})}{\partial x}|_{x=p_i}] \quad (2.16)$$

with $\epsilon_i = \pm 1$ and variables u_{2r} and Lagrange multipliers L_i, Q_i, p_i. In fact L_i, Q_i conditions tell us that there are l double roots at p_i as shown by the factor $(H_l(x))^2$.

From the equation (2.16) we first get

$$\begin{aligned} \frac{\partial}{\partial Q_i} : \quad & \frac{\partial(\frac{P_{2N}(x^2,u)}{x^2})}{\partial x}|_{x=p_i} = 0 \\ \frac{\partial}{\partial p_i} : \quad & Q_i\frac{\partial^2(\frac{P_{2N}(x^2,u)}{x^2})}{\partial x^2}|_{x=p_i} = 0. \end{aligned}$$

Since in general $\frac{\partial^2(\frac{P_{2N}(x^2,u)}{x^2})}{\partial x^2}$ is not degenerate, we get $Q_i = 0$. Using this result we get

$$\begin{aligned} \frac{\partial}{\partial u_{2r}} : \quad & g_{2r} + \sum_{i=1}^{l}\sum_{j=0}^{N} L_i p_i^{2N-2j-2}\frac{\partial s_{2j}}{\partial u_{2r}} = 0, \\ & \to g_{2r} = \sum_{i=1}^{l}\sum_{j=0}^{N} L_i p_i^{2N-2j-2}s_{2j-2r} \end{aligned}$$

where we have used the expansion $P_{2N}(x^2,u) = \sum_{r=0}^{N} s_{2r}x^{2N-2r}$ with $s_0 = 1$ and $\frac{\partial s_{2j}}{\partial u_{2r}} = -s_{2j-2r}$. Because the SW curve is an even function of x, roots p_i must be in pairs as $(p_i, -p_i)$ and we can write the sum as

$$g_{2r} = \sum_{i=1}^{l/2}\sum_{j=0}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}s_{2j-2r} \quad (2.17)$$

where we have assumed that l is even number.

Now we calculate

$$\begin{aligned} W'(x) &= \sum_{r=1}^{N} g_{2r}x^{2r-1} \\ &= \sum_{r=1}^{N}\sum_{i=1}^{l/2}\sum_{j=0}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}s_{2j-2r}x^{2r-1} \\ &= \sum_{r=-\infty}^{N}\sum_{i=1}^{l/2}\sum_{j=0}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}s_{2j-2r}x^{2r-1} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \end{aligned}$$

where the $L \equiv \sum_{i=1}^{l/2}(L_{i+} + L_{i-})\epsilon_i$. It can be shown by taking $\frac{\partial}{\partial L_i}$ of (2.16) that $\epsilon_i = \epsilon_{i+} = \epsilon_{i-}$. Replacing $\sum_{j=0}^{N}$ by $\sum_{j=-\infty}^{N}$ since these terms are of higher order, we get

$$\begin{aligned} W'(x) &= \sum_{r=-\infty}^{N}\sum_{i=1}^{l/2}\sum_{j=-\infty}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}s_{2j-2r}x^{2r-1} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \\ &= \sum_{i=1}^{l/2}\sum_{j=-\infty}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}x^{-2N-1+2j}\sum_{\widetilde{r}\equiv j-r:j-N}^{+\infty} s_{2\widetilde{r}}x^{2N-2\widetilde{r}} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \\ &= P_{2N}(x^2, u)\sum_{i=1}^{l/2}\sum_{j=-\infty}^{N}(L_{i+} + L_{i-})p_i^{2N-2j-2}x^{-2N-1+2j} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \\ &= P_{2N}(x^2, u)\sum_{i=1}^{l/2}\frac{(L_{i+} + L_{i-})}{xp_i^2}\frac{1}{1-\frac{p_i^2}{x^2}} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \\ &= xP_{2N}(x^2, u)\sum_{i=1}^{l/2}\frac{(L_{i+} + L_{i-})}{p_i^2}\frac{1}{x^2 - p_i^2} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \ . \end{aligned}$$

With the definition

$$\sum_{i=1}^{l/2}\frac{(L_{i+} + L_{i-})}{p_i^2}\frac{x^2}{x^2 - p_i^2} = \frac{B_l(x)}{H_l(x)}$$

we have

$$W'(x) = B_l(x)\frac{P_{2N}(x^2, u)}{xH_l(x)} - 2L\Lambda^{2N-2}x^{-1} + \mathcal{O}(x^{-3}) \ . \tag{2.18}$$

From this we can write

$$W'(x) + 2L\Lambda^{2N-2}x^{-1} = B_l(x)\sqrt{F_{4N-2l-2}(x) + \frac{4\Lambda^{4N-4}x^2}{H_l(x)^2}} + \mathcal{O}(x^{-3}) \ .$$

Comparing the degree at two sides we find $deg(B_l) = 2n + 1 - (2N - l - 1) \geq 0$. If we set $l = 2N - 2n - 2$, $B_l(x) = g_{2n+2}$ is just a constant, so finally we get the wanted relationship

$$g_{2n+2}^2 F_{2(2n+1)}(x) = W'^2(x) + 4L\Lambda^{2N-2}g_{2n+2}x^{2n} + ... = W'^2(x) + f_{2n}(x) \tag{2.19}$$

Furthermore from the form (2.18), it can be seen that both $H_l(x)$ and $F_{4N-2l-2}(x)$ are functions of x^2.

There is another important relationship we can get. Differentiating (2.16) regarding to $\log(\Lambda^{4N-4})$, we have

$$\begin{aligned} \frac{dW}{d\log(\Lambda^{4N-4})} &= \frac{\partial W}{\partial \log(\Lambda^{4N-4})} + \frac{\partial W}{\partial u_{2r}}\frac{\partial u_{2r}}{\partial \log(\Lambda^{4N-4})} \\ &\quad + \frac{\partial W}{\partial L_i}\frac{\partial L_i}{\partial \log(\Lambda^{4N-4})} + \frac{\partial W}{\partial p_i}\frac{\partial p_i}{\partial \log(\Lambda^{4N-4})} \\ &= \frac{\partial W}{\partial \log(\Lambda^{4N-4})} = -\sum_{i}^{l} L_i\epsilon_i\Lambda^{2N-2} \\ &= -L\Lambda^{2N-2} \end{aligned}$$

where in the third line we have used equations for u_{2r}, p_i, L_i and in the fourth line, the definition of L. From (2.19) we can read out the leading coefficient of $f_{2n}(x)$ to be $b_{2n} = 4L\Lambda^{2N-2}g_{2n+2}$, so we get

$$\frac{dW}{d\log(\Lambda^{4N-4})} = -\frac{b_{2n}}{4g_{2n+2}} \tag{2.20}$$

which has been advertised in (2.13).

2.2.1. The $SO(2N+1)$ and $Sp(2N)$ cases

Having done the case of $SO(2N)$ in detail, we will just scratch the $SO(2N+1)$ and $Sp(2N)$ cases. For $SO(2N+1)$, we write the factorized SW curve as

$$(\frac{P_{2N}(x^2,u)}{x})^2 - 4\Lambda^{4N-2} = (H_l(x))^2 F_{4N-2l-2}(x), \tag{2.21}$$

and the corresponding low energy effective action

$$W = \sum_{r=1}^{n+1} g_{2r}u_{2r} + \sum_{i=1}^{l}[L_i((\frac{P_{2N}(x^2,u)}{x})|_{x=p_i} - 2\epsilon_i\Lambda^{2N-1}) + Q_i\frac{\partial(\frac{P_{2N}(x^2,u)}{x})}{\partial x}|_{x=p_i}] \tag{2.22}$$

Using equations of Q_i, L_i, p_i, u_{2r} we get

$$g_{2r} = \sum_{i=1}^{l/2}\sum_{j=0}^{N}(L_{i+} - L_{i-})p_i^{2N-2j-1}s_{2j-2r}$$

and

$$\begin{aligned} W'(x) &= \sum_{r=1}^{N} g_{2r}x^{2r-1} \\ &= \sum_{r=-\infty}^{N}\sum_{i=1}^{l/2}\sum_{j=0}^{N}(L_{i+} - L_{i-})p_i^{2N-2j-1}s_{2j-2r}x^{2r-1} - 2L\Lambda^{2N-1}x^{-1} + \mathcal{O}(x^{-3}) \\ &= xP_{2N}(x^2,u)\sum_{i=1}^{l/2}\frac{(L_{i+} - L_{i-})}{p_i}\frac{1}{x^2 - p_i^2} - 2L\Lambda^{2N-1}x^{-1} + \mathcal{O}(x^{-3}) \end{aligned}$$

with the definition $L \equiv \sum_{i=1}^{l/2}(L_{i+} - L_{i-})\epsilon_i$. Defining

$$\sum_{i=1}^{l/2}\frac{(L_{i+} - L_{i-})}{p_i}\frac{x^2}{x^2 - p_i^2} = \frac{B_l(x)}{H_l(x)}$$

we can write

$$W'(x) + 2L\Lambda^{2N-1}x^{-1} = B_l(x)\sqrt{F_{4N-2l-2}(x) + \frac{4\Lambda^{4N-2}}{H_l(x)^2}} + \mathcal{O}(x^{-3}).$$

Setting $l = 2N - 2n - 2$, $B = g_{2n+2}$ we finally get

$$g_{2n+2}^2 F_{2(2n+1)}(x) = W'^2(x) + 4L\Lambda^{2N-1} g_{2n+2} x^{2n} + ... = W'^2(x) + f_{2n}(x) \tag{2.23}$$

Again, differentiating (2.22) by $\log(\Lambda^{4N-2})$, we get

$$\frac{dW}{d\log(\Lambda^{4N-2})} = -\sum_i^l L_i \epsilon_i \Lambda^{2N-1} = -L\Lambda^{2N-1} = -\frac{b_{2n}}{4g_{2n+2}} \tag{2.24}$$

For the $Sp(2N)$ gauge group we write down

$$[x^2 P_{2N}(x^2, u) + 2\Lambda^{2N+2}]^2 - 4\Lambda^{4N+4} = (H_l(x))^2 F_{4N-2l+2}(x) x^2, \tag{2.25}$$

and

$$W = \sum_{r=1}^{n+1} g_{2r} u_{2r} + \sum_{i=1}^{l} [L_i (x^2 P_{2N}(x^2, u)|_{x=p_i} - 2\epsilon_i \Lambda^{2N+2}) + Q_i \frac{\partial(x^2 P_{2N}(x^2, u))}{\partial x}|_{x=p_i}] \tag{2.26}$$

where $\epsilon_i = 0, -2$ which is different from the SO case. From the W, we find

$$g_{2r} = \sum_{i=1}^{l} \sum_{j=0}^{N} L_i p_i^{2N-2j+2} s_{2i-2r} = \sum_{i=1}^{l/2} \sum_{j=0}^{N} (L_{i+} + L_{i-}) p_i^{2N-2j+2} s_{2i-2r}$$

and

$$\begin{aligned} W'(x) &= \sum_{r=1}^{N} g_{2r} x^{2r-1} \\ &= \sum_{r=-\infty}^{N} \sum_{i=1}^{l/2} \sum_{j=0}^{N} (L_{i+} + L_{i-}) p_i^{2N-2j+2} s_{2j-2r} x^{2r-1} - 2L\Lambda^{2N+2} x^{-1} + \mathcal{O}(x^{-3}) \\ &= x P_{2N}(x^2, u) \sum_{i=1}^{l/2} (L_{i+} + L_{i-}) p_i^2 \frac{1}{x^2 - p_i^2} - 2L\Lambda^{2N+2} x^{-1} + \mathcal{O}(x^{-3}) \end{aligned}$$

with the definition $L \equiv \sum_{i=1}^{l/2} (L_{i+} + L_{i-}) \epsilon_i$. Writing

$$\sum_{i=1}^{l/2} (L_{i+} + L_{i-}) p_i^2 \frac{1}{x^2 - p_i^2} = \frac{B_{l-2}(x)}{H_l(x)}$$

we get

$$\begin{aligned} W'(x) &= B_{l-2}(x) \frac{1}{x} \frac{x^2 P_{2N}(x^2, u)}{H_l(x)} - 2L\Lambda^{2N+2} x^{-1} + \mathcal{O}(x^{-3}) \\ &= B_{l-2}(x) \frac{1}{x} (\sqrt{F_{4N-2l+2}(x) x^2 + \frac{4\Lambda^{4N+4}}{H_l(x)^2}} - \frac{2\Lambda^{2N+2}}{H_l(x)}) - 2L\Lambda^{2N+2} x^{-1} + \mathcal{O}(x^{-2}) \\ &= B_{l-2}(x) (\sqrt{F_{4N-2l+2}(x) + \frac{4\Lambda^{4N+4}}{x^2 H_l(x)^2}} - \frac{2\Lambda^{2N+2}}{x H_l(x)}) - 2L\Lambda^{2N+2} x^{-1} + \mathcal{O}(x^{-2}) \end{aligned}$$

Setting $l = 2N - 2n$ and $B_{l-2}(x) = g_{2n+2}$, we have

$$g^2_{2n+2} F_{2(2n+1)} = W'(x)^2 + 4L\Lambda^{2N+2} g_{2n+2} x^{2n} + ... = W'(x)^2 + f_{2n}(x) \tag{2.27}$$

Differentiating (2.26) by $\log(\Lambda^{4N+4})$ we found

$$\frac{dW}{d\log(\Lambda^{4N+4})} = -\sum_i^l L_i \epsilon_i \Lambda^{2N+2} = -L\Lambda^{2N+2} = -\frac{b_{2n}}{4g_{2n+2}} \tag{2.28}$$

2.3. The $\Lambda \to 0$ limit

To compare with the geometric picture, we need to discuss the solution in field theory at the limit $\Lambda \to 0$. In this limit, the factorization is easy to be solved. For example, for $SO(2N)$ gauge groups, we propose that

$$\begin{aligned} [P_{2N}(x,u)]^2 &= [x^{2N_0} \prod_{i=1}^n (x^2 + a_i^2)^{N_i}]^2 \\ &= [\frac{W'(x)^2}{g^2_{2n+2}}] x^2 [x^{2N_0-2} \prod_{i=1}^n (x^2 + a_i^2)^{N_i - 1}]^2 \end{aligned}$$

where the first line tells us that the proposed $P_{2N}(x,u)$ does satisfy the boundary condition. From this factorized form we can read out that $f_{2n}(x) = 0$. In this limit the effective action is calculated as

$$\begin{aligned} W_{eff} &= \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} \mathrm{Tr}[\Phi^{2r}] = \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} \sum_{i=-n}^{n} (-)^r N_i a_i^{2r} \\ &= \sum_{i=-n}^{n} N_i \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} (-)^r a_i^{2r} = \sum_{i=-n}^{n} N_i W(\alpha_i) \end{aligned} \tag{2.29}$$

where we have used the from $\Phi = diag(0_{2N_0}, (ia_i, -ia_i)_{N_i})$ and α_i are these eigenvalues.

Similar calculations can be done for other two gauge groups as

$$\begin{aligned} SO(2N+1): \quad & [P_{2N}(x,u)]^2 = [x^{2N_0} \prod_{i=1}^n (x^2 + a_i^2)^{N_i}]^2 \\ & = [\frac{W'(x)^2}{g^2_{2n+2}}] x^2 [x^{2N_0-2} \prod_{i=1}^n (x^2 + a_i^2)^{N_i - 1}]^2, \\ Sp(2N): \quad & x^4 [P_{2N}(x,u)]^2 = x^4 [x^{2N_0} \prod_{i=1}^n (x^2 - a_i^2)^{N_i}]^2 \\ & = [\frac{W'(x)^2}{g^2_{2n+2}}] x^2 [x^{2N_0} \prod_{i=1}^n (x^2 + a_i^2)^{N_i - 1}]^2, \end{aligned}$$

with $f_{2n}(x) = 0$. The effective action of $SO(2N+1)$ is same as $SO(2N)$, but for $Sp(2N)$ it is modified to

$$W_{eff} = \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} Tr[\Phi^{2r}] = \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} \sum_{i=-n}^{n} N_i a_i^{2r}$$

$$= \sum_{i=-n}^{n} N_i \sum_{r=1}^{n+1} \frac{g_{2r}}{2r} a_i^{2r} = \sum_{i=-n}^{n} N_i W(\alpha_i)$$

where $\Phi = diag(0_{2N_0}, (a_i, -a_i)_{N_i})$.

3. The geometric picture

The geometric duals to field theories are given in [17], where it was conjectured that low energy (holomorphic) dynamics can be calculated by geometric dual theories. Later in [20], this conjecture has been proved for $U(N)$ gauge groups with one adjoint field Φ. The geometric dual theories have been generalized from $U(N)$ gauge groups to SO/Sp gauge groups in [32] and explicit examples to support this conjecture were given in [34]. It is our aim in this paper to give a proof for SO/Sp gauge groups.

3.1. Review

To have the geometric dual theory, first we need to geometrically engineer the $\mathcal{N} = 2$ field theory deformed by superpotential (2.1). It can be done by wrapping D5-branes along two cycles in the non-compact, nontrivial fibrated Calabi-Yau threefold

$$uv + w^2 + W'(x)^2 = 0, \qquad W'(x) = g_{2n+2} x \prod_{j=1}^{n} (x^2 \pm a_i^2), \tag{3.1}$$

At each root of $W'(x)$ there is a blown up S^2 with N_i D5-branes wrapped around this S^2. The geometric dual theory is obtained via the geometric transition [35, 36] where S^2's are blown down and S^3's are blown up. At the same time, N_i D5-branes wrapped around S_i^2 disappear and are replaced by N_i units of H_{RR} fluxes through the new nontrivial S_i^3. The transition to S^3's corresponds to a complex deformation of the geometry as

$$uv + w^2 + W'(x)^2 + f_{2n}(x) = 0, \tag{3.2}$$

From this, we can calculate the effective superpotential in the geometric dual theory by

$$\frac{-1}{2\pi i} W_{eff} = \int_{CY} H \wedge \Omega = \sum_{i=-n}^{n} \int_{A_i} H \int_{B_i} \Omega - \int_{B_i} H \int_{A_i} \Omega \tag{3.3}$$

where $H = H_{RR} - \tau_{IIB} H_{NS}$, Ω the holomorphic three form on the CY 3-fold and A_i, B_i the symplectic bases.

As did in [20] we can reduce the integration on the CY 3-fold to the integration on the reduced surface

$$y^2 = W'(x)^2 + f_{2n}(x) \tag{3.4}$$

with reduced one forms h and $dx\lambda_{eff}$

$$dx\lambda_{eff} = dx\sqrt{W'(x)^2 + f_{2n}(x)}, \tag{3.5}$$

$$h = \int_{S^2} H, \tag{3.6}$$

so the effective action is simplified to

$$\frac{-1}{2\pi i}W_{eff} = \int_{\Gamma} h \wedge dx\lambda_{eff} = \sum_{i=-n}^{n} \int_{a_i} h \int_{b_i} dx\lambda_{eff} - \int_{b_i} h \int_{a_i} dx\lambda_{eff} \tag{3.7}$$

with a_i these compact one cycles and b_i, these corresponding non-compact dual one cycles.

When we discuss the SO/Sp gauge groups, we need to add the orientifold into the geometry[37, 32, 38]. The orientifold action will have following contributions. Firstly it contributes H_{RR} fluxes to the integration along the cycle around it. Secondly it pairs blown up S^3's except the one fixed by the orientifold action in CY 3-fold. In other words, the orientifold action requires the deformation $f_{2n}(x)$ to be a function of x^2.

Above discussions provide us with a convenient way to look at the problem. We can work first at the double covering space, where the result of $U(2N)$ can be applied, with the condition that it preserves the Z_2 action of orientifold. Then by putting the Z_2 action back, we get the results for SO/Sp gauge groups.

First let us discuss the choice of cycles in (3.7). These cycles are almost the same as these in [20] with only one extra condition that they preserve the Z_2 symmetry (see Figure 1). Since branch cuts are symmetric with one located at the center, it is not difficult to show

$$\oint_{\alpha_k} dx\lambda_{eff} = \oint_{\alpha_{-k}} dx\lambda_{eff}, \qquad \int_{C_k} dx\lambda_{eff} = \int_{C_{-k}} dx\lambda_{eff} \tag{3.8}$$

For the second equation, it is worth to notice that $C_k - C_{-k} = \sum_{j=-k,j\neq 0}^{k} \beta_j$ and $\oint_{\beta_j} dx\lambda_{eff} = -\oint_{\beta_{-j}} dx\lambda_{eff}$.

Now we can identify cycles

$$\int_{a_k} = \frac{1}{2}\frac{1}{2\pi i}\oint_{\alpha_k}, \qquad \int_{b_k} = \frac{1}{2\pi i}\int_{C_k},$$

and following the physical quantities

$$S_k = \int_{a_k} dx\lambda_{eff} = \frac{1}{2}\frac{1}{2\pi i}\oint_{\alpha_k} dx\lambda_{eff}, \tag{3.9}$$

$$\Pi_k = \frac{1}{2}\int_{b_k} dx\lambda_{eff} = \frac{1}{4\pi i}\int_{C_k} dx\lambda_{eff} = \frac{1}{2\pi i}\int_{a_k}^{\Lambda_0} dx\lambda_{eff}. \tag{3.10}$$

Among these variables in (3.9) and (3.10), (3.8) indicates that

$$S_k = S_{-k}, \qquad \Pi_k = \Pi_{-k}. \tag{3.11}$$

Furthermore, using the fact that D5-branes have been replaced by fluxes, we get

$$\int_{a_k} h = \frac{1}{2}N_i, (k \neq 0), \qquad \int_{a_0} h = \frac{1}{2}2\hat{N}_0, \qquad \int_{b_k} h = 2\tau_{YM}\ . \tag{3.12}$$

There are several things to be remarked. First, because of the orientifold plane, every D5-brane in the covering space carries only half of physical brane charges. Secondly, the

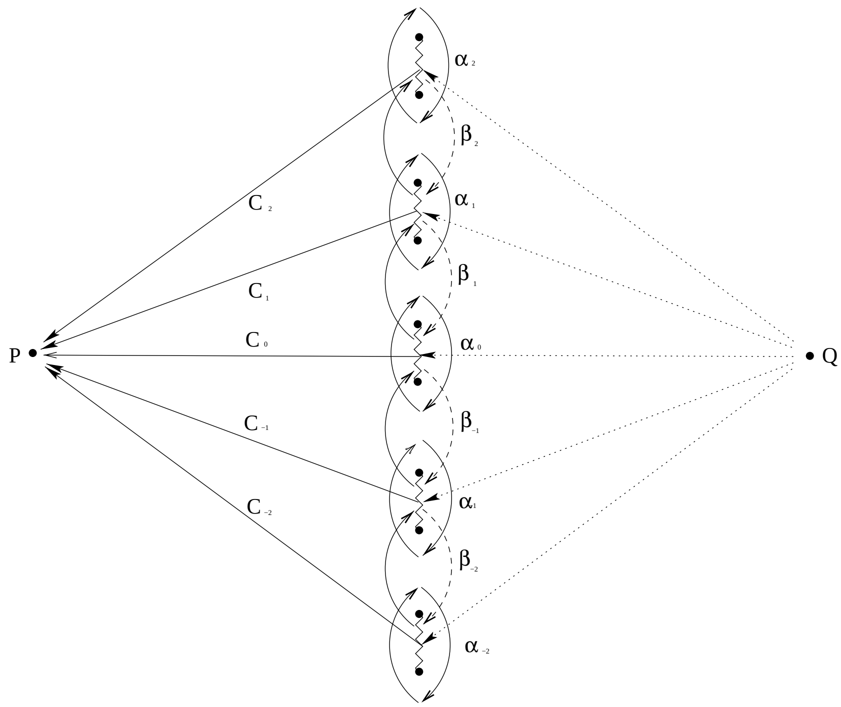

Figure 1: The choice of our cycles α_i, C_i, β_i. Notice that they are drawn in symmetric fashion. The solid line means that it is at the upper plane while the dotted line, the lower plane. Q is same point as P, but at lower plane.

physical brane charges of orientifold planes are $O5^- = -1$, $O5^+ = +1$ and $\widetilde{O5}^- = -1/2$, so $2N_0$, which mean initially total $2N_0$ D5-branes wrapped around the origin, are modified to $2N_0 - 2$ for $SO(2N)$ group, $2N_0 - 1$ for $SO(2N+1)$ group, and $2N_0 + 2$ for $Sp(2N)$ group. Thirdly, $\int_{b_k} h$ are independent of k, thus we require

$$\int_{C_k - C_l} h = 0 \Leftrightarrow \oint_{\beta_j} h = 0, \forall j \ . \tag{3.13}$$

Fourthly, summing all α_k contours together, we get

$$\oint_P \frac{h}{2\pi i} = \sum_{k=-n}^{n} \oint_{\alpha_k} \frac{h}{2\pi i} = \sum_{k=-n}^{n} N_i = 2\hat{N}, \qquad \oint_Q \frac{h}{2\pi i} = -2\hat{N} \tag{3.14}$$

where $2\hat{N}$ is $2N-2$ for $SO(2N)$ gauge group, $2N-1$, $SO(2N+1)$ gauge group and $2N+2$, $Sp(2N)$ gauge group. Later, we will find an 1-form h on the surface (3.4) satisfied both (3.13) and (3.14). Finally putting every thing together we can write the effective action as

$$\begin{aligned} \frac{-1}{2\pi i} W_{eff} &= \sum_{i=-n}^{n} \int_{a_i} h \int_{b_i} dx \lambda_{eff} - \int_{b_i} h \int_{a_i} dx \lambda_{eff} \\ &= \sum_{i=-n}^{n} \frac{1}{2} N_i (2\Pi_i) - 2\tau_{YM} S_i \\ &= 2\hat{N}_0 \Pi_0 + (\sum_{i=1}^{n} 2N_i \Pi_i) - 2\tau_{YM}(S_0 + 2\sum_{i=1}^{n} S_i) \end{aligned} \tag{3.15}$$

where in the last line, we keep cycle integrations of upper half planes only.

3.2. Some properties of W_{eff}

Now let us discuss some properties of the effective action W_{eff}. First if we let $\Lambda_0 \to e^{2\pi i}\Lambda_0$ at the upper plane, for every C_k, it will anti-clockwised enclose all brunch cuts, so

$$\Delta\Pi_k = \Delta(\frac{1}{2\pi i}\int_{a_k}^{\Lambda_0} dx\lambda_{eff}) = -\sum_{i=-n}^{+n}\oint_{\alpha_i}\frac{dx}{2\pi i}\lambda_{eff} = -2\sum_{i=-n}^{n} S_i,$$

which means that Π_k must depend on the cutoff Λ_0 as

$$\Pi_k = \frac{-2}{2\pi i}\sum_{i=-n}^{n} S_i \log\Lambda_0 + ... \tag{3.16}$$

We can also find the Λ_0 dependence directly by calculating

$$\begin{aligned}\Pi_k &= \frac{1}{2\pi}\int_{a_k}^{\Lambda_0} dx\lambda_{eff} = \frac{1}{2\pi i}\int_{a_k}^{\Lambda_0} dx\sqrt{W'(x)^2 + f_{2n}(x)} \\ &\sim \frac{1}{2\pi i}\int_{a_k}^{\Lambda_0} dx(W'(x) + \frac{b_{2n}}{2g_{2n+2}}\frac{1}{x} + \mathcal{O}(\frac{1}{x^2})) \\ &\sim \frac{1}{2\pi i}(W(\Lambda_0) + \frac{b_{2n}}{2g_{2n+2}}\log\Lambda_0 + \mathcal{O}(\frac{1}{\Lambda_0^2}))\end{aligned}$$

From these two expressions we get a very important relationship

$$\frac{-b_{2n}}{4g_{2n+2}} = \sum_{i=-n}^{n} S_i \tag{3.17}$$

In fact, this result can be obtained by summing all α_k cycles on the upper plane and push them to go around point P

$$\begin{aligned}\sum_{i=-n}^{n} S_i &= \sum_{i=-n}^{+n}\oint_{\alpha_i}\frac{1}{4\pi i}\lambda_{eff} = \frac{1}{4\pi i}\oint_P dx\sqrt{W'(x)^2 + f_{2n}(x)} \\ &= \frac{1}{4\pi i}\oint_P dx(W'(x) + \frac{f_{2n}(x)}{2W'(x)} + ...) \\ &= \frac{1}{4\pi i}\oint_P dx\frac{b_{2n}}{2g_{2n+2}x} = \frac{-b_{2n}}{4g_{2n+2}}\end{aligned}$$

Notice that at the last step, we integrate around the point at infinite.

Now we put (3.16) back into the effective action and get

$$W_{eff} = 2(\sum_{i=-n}^{n} S_i)[2\hat{N}\log\Lambda_0 + 2\pi i\tau_{YM}] + ...$$

Absorbing the cutoff Λ_0 into the physical scale Λ by $2\hat{N}\log\Lambda_0 + 2\pi i\tau_{YM} = 2\hat{N}\log\Lambda$, we get an important result

$$\frac{dW_{eff}}{d\log\Lambda^{4\hat{N}}} = \sum_{i=-n}^{n} S_i = \frac{-b_{2n}}{4g_{2n+2}} \tag{3.18}$$

This equation is same as (2.20), (2.24) and (2.28) got by calculations in the field theory if the $f_{2n}(x)$ in the field theory side is identified to the one in the dual geometry. We will show it is true.

Using the result (3.17) we can rewrite (3.15) into

$$\frac{-1}{2\pi i}W_{eff} = \sum_{k=-n}^{n} N_k \int_{a_k}^{\Lambda_0} \frac{dx}{2\pi i}\lambda_{eff} + \frac{b_{2n}\tau_{YM}}{2g_{2n+2}}$$

In the $\Lambda \to 0$ limit, $f_{2n}(x) = 0$ as well as $b_{2n} = 0$. Thus we have

$$W_{eff} = -\sum_{k=-n}^{n} N_k \int_{a_k}^{\Lambda_0} dx W'(x) = \sum_{k=-n}^{n} N_k W(a_k) - 2\hat{N}W(\Lambda_0) \tag{3.19}$$

It is equal to the result (2.29) in the field theory up to a constant[3].

Just like [20], here we have shown that for SO/Sp gauge groups, the W_{eff}^{f} calculated in the field theory and the W_{eff}^{G} calculated in the dual geometry have the same value at the classical limit $\Lambda \to 0$ and follow the same differential equation with respect to Λ, so they must be the same. These results finish the first step of proofs that the field theory is equivalent to the dual geometry.

3.3. The h and Related Seiberg-Witten curve

To show the equivalence between the field theory and the dual geometry we need to find the one form h satisfied conditions (3.13) and (3.14). To do so, first we rewrite the effective action as

$$\begin{aligned}
\frac{-2}{2\pi i}W_{eff} &= \sum_{k=-n}^{n} \oint_{\alpha_k} \frac{h}{2\pi i} \int_{C_k} \frac{dx\lambda_{eff}}{2\pi i} - \oint_{\alpha_k} \frac{dx\lambda_{eff}}{2\pi i} \int_{C_k} \frac{h}{2\pi i} \\
&= 2\hat{N}\int_{C_0} \frac{dx\lambda_{eff}}{2\pi i} + \frac{b_{2n}\tau_{YM}}{2g_{2n+2}} - \sum_{k=1}^{n} 2N_k\left(\sum_{j=1}^{k} \oint_{\beta_j} \frac{dx\lambda_{eff}}{2\pi i}\right) .
\end{aligned}$$

To reach the last step, we have used following facts $C_{-k} = \sum_{j=1}^{k} \beta_{-k} + C_0$, $C_k = -\sum_{j=1}^{k} \beta_k + C_0$ and (3.13), (3.14). From this we get equations of motion

$$\frac{-2}{2\pi i}\frac{\partial W_{eff}}{\partial b_{2l}} = 2\hat{N}\int_{C_0} \frac{dx}{2\pi i}\frac{\partial \lambda_{eff}}{\partial b_{2l}} + \frac{\tau_{YN}}{2g_{2n+2}}\delta_{l,n} - \sum_{k=1}^{n} 2N_k\left(\sum_{j=1}^{k} \oint_{\beta_j} \frac{dx}{2\pi i}\frac{\partial \lambda_{eff}}{\partial b_{2l}}\right) \tag{3.20}$$

[3]There is a small difference between (2.29) and (3.19). In (3.19) N_0 are modified to $\hat{N}_0$. However, in $\Lambda \to 0$ limit, $a_0 = 0$ and $W(a_0) = 0$ so this difference does not effect anything.

For $l=n$, since $\frac{\partial \lambda_{eff}}{\partial b_{2n}} \rightarrow \frac{1}{2g_{2n+2}}\frac{dx}{x}$ at large x limit, the first two terms give $\frac{2\hat{N}\log\Lambda_0+2\pi i\tau_{YM}}{g_{2n+2}}$ with a cutoff Λ_0 and we need to introduce some Λ (depending on b_{2r}) to satisfy the equation.

For $l<n$, notice that

$$dx\frac{\partial \lambda_{eff}}{\partial b_{2l}} = \frac{x^{2l}dx}{2\sqrt{W'(x)^2+f_{2n}(x)}} = \frac{x^{2l}dx}{2\sqrt{F_{2(2n+1)}(x)}}$$
$$= \frac{t^l dt}{4\sqrt{t\widetilde{F}_{2n+1}(t)}}, \qquad t=x^2, \quad \widetilde{F}_{2n+1}(t)=F_{2(2n+1)}(x)$$

The equations of motion (3.20) can be rewritten as

$$\hat{N}\int_{C_0}\frac{t^l dt}{\widetilde{y}} = \sum_{k=1}^{n} N_k(\sum_{j=1}^{k}\oint_{\beta_j}\frac{t^l dt}{\widetilde{y}}), \quad \forall l \tag{3.21}$$

with

$$\Gamma: \quad \widetilde{y}^2 = t\widetilde{F}_{2n+1}(t)\ . \tag{3.22}$$

The curve (3.22) is, in fact, the related Seiberg-Witten curve after the Z_2 quotient from the covering space [33]. Its genus is n which corresponds to the fact there are N $U(1)$ left in the field theory. The integrand $\frac{u^l du}{\widetilde{y}}$ $l=0,...,n-1$ are bases of holomorphic one forms on Γ and equations (3.21) mean that the left hand side is zero up to some periods on Γ. According to Abel's theorem, there must be a meromorphic function on Γ with divisor $\hat{N}[P-Q]$ [4]. Furthermore, h is a holomorphic one form on Γ with certain properties.

Now we have translated field theory equations into the existence of a particular Riemann surface Γ. We will show that if the factorization form holds, the particular Riemann surface exists. Let us start with $SO(2N)$ gauge group. Rewriting

$$(W'(x)^2+f_{2n}(x))x^2H_{2N-2n-2}(x)^2 = g_{2n+2}^2(P_{2N}(x,u)^2-4\gamma^2x^4) \tag{3.23}$$

with the boundary condition that $P_{2N}(x,u)|_{\gamma\rightarrow 0} = x^{2N_0}\prod_{i=1}^{n}(x^2+a_i^2)^{N_i}$ as

$$\widetilde{H}_{N-n-1}(t)^2\widetilde{y}^2 = g_{2n+2}^2(\widetilde{P}_N(t,u)^2-4\gamma^2t^2)$$

and defining

$$zt = \widetilde{P}_N(t,u) - \frac{1}{g_{2n+2}}\widetilde{y}(t)\widetilde{H}_{N-n-1}(t), \tag{3.24}$$

we get the equation satisfied by z

$$z - \frac{2\widetilde{P}_N(t,u)}{t} + \frac{4\gamma^2}{z} = 0 \tag{3.25}$$

[4] It maybe a little confusing that for $SO(2N+1)$ gauge group we have $\hat{N}=N-1/2$ not integer. The reason is that from the brane picture, there is a stuck D5-brane on top of the orientifold without the image, so the best way to discuss $SO(2N+1)$ is in the covering space.

(please notice that $\frac{2\widetilde{P}_N(t,u)}{t}=\frac{2P_{2N}(x,u)}{x^2}$, so it is the polynomial of t). Notice that at this moment the γ is an undetermined parameter which will be shown to be the dynamical scale in the Seiberg-Witten curve by independent derivations. From (3.25), we see immediately that z has zeros of order $N-1$ at P and poles of order $N-1$ at Q and holomorphic elsewhere. Thus

$$h=\frac{-dz}{z} \tag{3.26}$$

satisfies all conditions required by the geometry. To check this we lift to the covering space by replacing $t=x^2$. As noticed in [20], integration around α_k cycles does not depend on γ, so we can evaluate them by setting $\gamma\to 0$

$$\begin{aligned}\oint_{\alpha_i}\frac{1}{2\pi i}h &= \frac{1}{2\pi i}\oint_{\alpha_i}\frac{-dz}{z}=\frac{1}{2\pi i}\oint_{\alpha_i}-d(\log z)\\ &= \frac{1}{2\pi i}\oint_{\alpha_i}d(\frac{2P_{2N}(x,u)}{x^2}|_{\gamma\to 0})=N_i\frac{1}{2\pi i}\oint_{\alpha_i}-d(\log(x-ia_i))\\ &= N_i \quad for\ (i\neq 0), \qquad or\ (2N_0-2),\ i=0\end{aligned}$$

where we have used the boundary condition $P_{2N}(x,u)|_{\gamma\to 0}=x^{2N_0}\prod_{i=1}^n(x^2+a_i^2)^{N_i}$ and the direction of cycles is clockwise. Furthermore

$$\int_{C_i-C_j}\frac{1}{2\pi i}\frac{-dz}{z}=0$$

since C_i-C_j cycles do not cross any branch cut of the logarithmic function. To determine the γ, we solve

$$z=\frac{P_{2N}(x,u)}{x^2}\pm\sqrt{(\frac{P_{2N}(x,u)}{x^2})^2-4\gamma^2} \tag{3.27}$$

and integrate directly

$$2\tau_{YM}=\int_{C_k}\frac{1}{2\pi i}h=\frac{2}{2\pi i}\int_{a_k^+}^{\Lambda_0}\frac{-dz}{z}=\frac{-2}{2\pi i}\log(z)|_{a_k^+}^{\Lambda_0}=\frac{-2}{2\pi i}\log\frac{2\Lambda_0^{2N-2}}{\pm 2\gamma}$$

where we have used the fact that at $x=a_k^+$, $W'(x)^2+f_{2n}(x)=0$, so by the factorization form we have

$$(\frac{P_{2N}(x,u)}{x^2})^2=4\gamma^2\ .$$

Because we have required $2\pi i\tau_{YM}+(2N-2)\log\Lambda_0=(2N-2)\log\Lambda$, it gives immediately

$$\pm\gamma=\Lambda^{2N-2}\ . \tag{3.28}$$

Results (3.23) and (3.28) prove that the complex deformation $f_{2n}(x)$ in the dual geometry is same as the $f_{2n}(x)$ in the field theory by factorization.

Similar calculations can be done for $SO(2N+1)$ and $Sp(2N)$ gauge groups. For $SO(2N+1)$, we take the factorized form

$$(W'(x)^2+f_{2n}(x))x^2H_{2N-2n-2}(x)^2=g_{2n+2}^2(P_{2N}(x,u)^2-4\gamma^2x^2) \tag{3.29}$$

with the boundary condition $P_{2N}(x,u)|_{\gamma\to 0} = x^{2N_0}\prod_{i=1}^{n}(x^2+a_i^2)^{N_i}$ and define z by

$$z - \frac{2P_{2N}(x,u)}{x} + \frac{4\gamma^2}{z} = 0 \tag{3.30}$$

. Notice that $\frac{2P_{2N}(x,u)}{x}$ does not have poles at $x=0$. It is easy to see that z has zeros of order $2N-1$ at P and poles of order $2N-1$ at Q (notice that now it is in the covering space). Defining h as in (3.26) and doing same calculations, it is easy to show that h satisfies all required conditions. Directly integrating h along any C_k, it can be seen that

$$\pm\gamma = \Lambda^{2N-1} \tag{3.31}$$

. For $Sp(2N)$ gauge group, we use

$$(W'(x)^2 + f_{2n}(x))x^2 H_{2N-2n}(x)^2 = g_{2n+2}^2[x^2 P_{2N}(x,u) + 2\gamma]^2 - 4\gamma^2 \tag{3.32}$$

with the boundary condition $P_{2N}(x,u)|_{\gamma\to 0} = x^{2N_0}\prod_{i=1}^{n}(x^2+a_i^2)^{N_i}$ and define z by the equation

$$z + \frac{4\gamma^2}{z} - 2(x^2 P_{2N}(x,u) + 2\gamma) = 0 \tag{3.33}$$

with zeros of order $2N+2$ at P and poles of order $2N+2$ at Q. Using h as in (3.26) it is easy to check all required conditions for h and determine

$$\pm\gamma = \Lambda^{2N+2} \tag{3.34}$$

3.4. The coupling constant matrix τ_{ij}

Now the last piece we need to do is to check that the coupling constant matrix

$$\tau_{ij} = \frac{\partial^2\mathcal{F}}{\partial S_i \partial S_j}$$

is indeed given by periods of the reduced Seiberg-Witten curve. Since we have shown that $S_k = S_{-k}, \Pi_k = \Pi_{-k}$, there are only $n+1$ independent S_k and Π_k which, for simplicity, can be chosen to be S_k, Π_k with $k = 0, 1, ..., n$ with relations

$$\Pi_k = \frac{\partial\mathcal{F}}{\partial S_k}, \quad k > 0, \qquad \Pi_0 = 2\frac{\partial\mathcal{F}}{\partial S_0}\ . \tag{3.35}$$

The reason for the second equation is that under the Z_2 action, S_0 is mapped to itself, so the physical glueball field $S_0^f = S_0/2$ and $\Pi_0 = \frac{\partial\mathcal{F}}{\partial S_0^f}$.

We define new bases of cycles as

$$\frac{\partial}{\partial\tilde{S}_0} = \frac{1}{4\hat{N}}[4\hat{N}_0\frac{\partial}{\partial S_0} + \sum_{j=1}^{n} 2N_j\frac{\partial}{\partial S_j}], \quad \frac{\partial}{\partial\tilde{S}_{i>1}} = \frac{\partial}{\partial S_i} - \frac{\partial}{\partial S_{i-1}}, \quad \frac{\partial}{\partial\tilde{S}_1} = \frac{\partial}{\partial S_1} - 2\frac{\partial}{\partial S_0}. \tag{3.36}$$

Then using equations of motion for the effective action (3.15)

$$\frac{\partial}{\partial S_k}[2\hat{N}_0\Pi_0 + (\sum_{i=1}^{n} 2N_i\Pi_i) - 2\tau_{YM}(S_0 + 2\sum_{i=1}^{n} S_i)] = 0 \tag{3.37}$$

we see immediately

$$\widetilde{\tau}_{00} = \frac{\partial^2 \mathcal{F}}{\partial \widetilde{S}_0^2} = \frac{2\tau_{YM}}{4\hat{N}}, \quad \widetilde{\tau}_{0,i\neq 0} = \frac{\partial^2 \mathcal{F}}{\partial \widetilde{S}_i \partial \widetilde{S}_0} = 0 \ .$$

In fact, $\widetilde{\tau}_{00}$ is the coupling constant of central $U(1)$ in double covering $U(2N)$ gauge group. When we project the $U(2N)$ to SO/Sp gauge groups by orientifold, the $U(1)$ is broken to global Z_2 symmetry as discussed in [33]. For other coupling constants

$$\widetilde{\tau}_{ij} = \frac{\partial^2}{\partial \widetilde{S}_i \partial \widetilde{S}_j}\mathcal{F} = \frac{\partial}{\partial \widetilde{S}_i}(\Pi_j - \Pi_{j-1}) = \frac{\partial}{\partial \widetilde{S}_i}\int_{C_j - C_{j-1}} \lambda_{eff}, \quad i, j \geq 1$$

by taking b_{2r} as new independent variables, we have

$$\begin{aligned}
\widetilde{\tau}_{ij} &= \sum_{r=0}^{n-1} \frac{\partial b_{2r}}{\partial \widetilde{S}_i}\frac{\partial}{\partial b_{2r}}(\int_{C_j - C_{j-1}} \lambda_{eff}) + \frac{\partial b_{2n}}{\partial \widetilde{S}_i}\frac{\partial}{\partial b_{2n}}(\int_{C_j - C_{j-1}} \lambda_{eff}) \\
&= \sum_{r=0}^{n-1} \frac{\partial b_{2r}}{\partial \widetilde{S}_i}\frac{\partial}{\partial b_{2r}}(\int_{C_j - C_{j-1}} \lambda_{eff})
\end{aligned} \tag{3.38}$$

where the second term drops out because $b_{2n} = -4g_{2n+2}(S_0 + 2\sum_{j=1}^{n} 2S_j)$ and $\frac{\partial b_{2n}}{\partial \widetilde{S}_i} = 0$. Using $\lambda_{eff} = \sqrt{W'(x)^2 + f_{2n}(x)}$, it is easy to see that

$$dx\frac{\partial \lambda_{eff}}{\partial b_{2r}} = dx\frac{x^{2r}}{2\lambda_{eff}} = dt\frac{t^r}{2\widetilde{y}(t)}, \quad r = 0, .., n-1$$

where $\widetilde{y}$ is given in (3.22) to be exactly the reduced Seiberg-Witten curve. Since $dt\frac{t^r}{2\widetilde{y}(t)}$ with $r = 0, 1, ..., n-1$ form a bases of holomorphic one forms on the reduced Riemann surface Γ and $\{\alpha_i, C_i - C_{i-1}\}$ form a basis for $H_1(\Gamma, Z)$, by (3.38) $\widetilde{\tau}_{ij}$ are indeed given by the period matrix of Γ. This completes the proof that the effective action and the coupling constants in the field theory can be equivalently calculated by the dual geometry using the large N duality.

Before closing this section, let us remark the role of z defined above. It can be shown that $x\frac{dz}{z}$ is exactly the Seiberg-Witten differential. For example, in the case of $SO(2N)$ gauge group, using (3.27) and $y^2 = P_{2N}(x^2, u)^2 - 4\Lambda^{4N-4}x^4$ we get

$$\begin{aligned}
x\frac{dz}{z} &= xdx(\frac{(\frac{P_{2N}(x,u)}{x^2})'}{\sqrt{(\frac{P_{2N}(x,u)}{x^2})^2 - 4\Lambda^{4N-4}}}) = \frac{xdx}{y}[x^2(\frac{P'_{2N}(x,u)}{x^2} - 2\frac{P_{2N}(x,u)}{x^3})] \\
&= \frac{xdx}{y}[P_{2N}(x,u) - \frac{1}{2}P_{2N}(x,u)\frac{(4\Lambda^{4N-4}x^4)'}{4\Lambda^{4N-4}x^4}]
\end{aligned}$$

which is indeed the Seiberg-Witten differential [31]. In fact, $\frac{dz}{z}$ is nothing else, but the eigenvalue distribution function in the corresponding gauge field theory as emphasized in [10, 44]. Furthermore, in the classical limit $\Lambda \to 0$, we have

$$\frac{dz}{z} = dx(1 - \frac{2}{x}) \ . \tag{3.39}$$

The term $\frac{2}{x}$ counts the contribution of the orientifold plane. It is rather strange that *even in the classical limit the theory knows the presence of the orientifold plane.*

4. The matrix model

Recalling the proof of matrix model conjecture for $U(N)$ gauge theory with superpotential $W(\Phi)$ given in [1], the first step is to show that from the corresponding matrix model, the spectral curve which is same as that in the dual geometry, can be derived. The second step is to match various integrations along compact and non-compact cycles at both sides (matrix model side and dual geometric side). The last step is to show that the relationship among these integrations are same at both sides. We will follow the same logic here for SO/Sp gauge groups.

The matrix models for the SO/Sp gauge groups have been proposed in [58, 63, 64][5] The partition function of the matrix model is given by

$$Z = \frac{1}{Vol(G)} \int d\Phi exp(-\frac{1}{g_s} \operatorname{Tr} W(\Phi)) \tag{4.1}$$

where Φ is in the adjoint representation of relative groups. The group measure has been given explicitly in [65, 63] for general matrices. For these models, Feynman diagrams are unoriented double line diagrams which reflect the nature of SO/Sp gauge groups. Going to the eigenvalue integration we get

$$Z \sim \int \prod_i d\lambda_i [\prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2][\prod_i \lambda_i^2]^s \exp(-\frac{2}{g_s} \sum_{i=1}^{M} \operatorname{Tr} W(\lambda_i)) \tag{4.2}$$

where $s = 0$ for $SO(2M)$ and $s = 1$ for $SO(2M+1)/Sp(M)$. Putting the Vandermonde determinant to the exponential we get

$$S(\lambda) = -\frac{2}{g_s} \sum_{i=1}^{M} \operatorname{Tr} W(\lambda_i) - \sum_{i<j} \log(\lambda_i^2 - \lambda_j^2)^2 - s \sum_{i=1}^{M} \log \lambda_i^2 \ . \tag{4.3}$$

Saddle point approximation of (4.3) gives us equations of motion of eigenvalues

$$\frac{1}{g_s} W'(\lambda_i) - 2\lambda_i \sum_{j \neq i} \frac{1}{\lambda_i^2 - \lambda_j^2} - \frac{s}{\lambda_i} = 0 \tag{4.4}$$

[5] In previous version, we follow the orthogonal and symplectic ensembles matrix model in [9]. However, from the point of view of field theory, it is more natural to use the matrix model proposed in [58, 63, 64]. Our treatment in this section will follow these three papers.

Define the resolvent to be

$$\omega(x) = \frac{-1}{M}\,\mathrm{Tr}\,\frac{1}{x-\Phi} = \frac{1}{M}\sum_{i=1}^{M}\frac{2x}{x^2-\lambda_i^2} \tag{4.5}$$

where we have used the fact that both $\pm\lambda_i$ are eigenvalues of Φ for SO/Sp gauge groups. With some algebraic operations we get

$$\omega(x)^2 - \frac{1}{M}[\omega'(x) - \frac{1-2s}{x}\omega(x)] - \frac{4}{\mu^2}f_{2n}(x) + \frac{2}{\mu}\omega(x)W'(x) = 0 \tag{4.6}$$

where

$$f_{2n}(x) = g_s\sum_{i=1}^{M}\frac{\lambda_i W'(\lambda_i) - xW'(x)}{x^2-\lambda_i^2} \tag{4.7}$$

and $\mu = g_s M$ which will be kept to be constant in the large M limit. Notice that since $W'(x)$ is an odd function of x, $f_{2n}(x)$ will be an even polynomial of x with degree $2n$. Also the difference between $SO(2M)$ and $SO(2M+1)/Sp(M)$ in (4.6) is counted by the $(1-2s)$ factor of $\mathcal{O}(M^{-1})$ order.

After taking the large M limit, differential equation (4.6) becomes algebraic equation

$$\omega(x)^2 - \frac{4}{\mu^2}f_{2n}(x) + \frac{2}{\mu}\omega(x)W'(x) = 0 \tag{4.8}$$

from which, if we define

$$y(x) = \frac{\mu}{2}(\omega(x) + \frac{W'(x)}{2}) \tag{4.9}$$

we get the spectral curve

$$y^2 = W'(x)^2 + f_{2n}(x) \tag{4.10}$$

Curve (4.10) is exactly same form (3.4) as in previous section. $y(x)$ is related to the force of moving eigenvalues away from their equilibrium positions as

$$y(\lambda) = \frac{g_s}{2}[\frac{\partial S(\lambda)}{\partial\lambda} + \frac{s}{\lambda}] \Longrightarrow \frac{g_s}{2}\frac{\partial S(\lambda)}{\partial\lambda}|_{large\ M} \tag{4.11}$$

Defining the eigenvalue distribution function as

$$\rho(\lambda) = \frac{1}{M}\sum_i \delta(\lambda-\lambda_i), \qquad \int d\lambda\rho(\lambda) = 1 \tag{4.12}$$

we have

$$\rho(\lambda) = \frac{1}{2\pi i}(\omega(\lambda+i0) - \omega(\lambda-i0)) \tag{4.13}$$

At large M limit, eigenvalues are clustered around different critical points given by the superpotential $W(\Phi)$ and filling factors can be calculated as

$$\frac{M_k}{M} = \oint_{\alpha_k} d\lambda\rho(\lambda) \tag{4.14}$$

Using the definition of y, we can get $\rho(\lambda) = \frac{1}{\pi i \mu}(y(\lambda + i0) - y(\lambda - i0))$, so

$$M_k = \frac{4M}{\mu} \oint_{\alpha_k} d\lambda \frac{y(\lambda)}{2\pi i} \Longrightarrow M_k = \frac{8}{g_s} S_k \tag{4.15}$$

by comparing with $S_i = \frac{1}{2} \oint_{\alpha_i} d\lambda \frac{1}{2\pi i} y(\lambda)$. Now changing filling factors by the amount ΔM_i, the action is changed to

$$\Delta F_{matrix} = \Delta M_i \int_{C_i/2} \frac{y(x)}{g_s} = \frac{32\pi i}{g_s^2} \Delta S_i \int_{C_i/2} \frac{y(x)}{2\pi i} = \frac{32\pi i}{g_s^2} \Delta S_i \Pi_i$$

and we get $\frac{\partial F_{matrix}}{\partial S_i} = \frac{32\pi i}{g_s^2} \Pi_i$. So to match results in the dual geometry, we just need to identify

$$F_{matrix} = \frac{32\pi i}{g_s^2} F_{field} \tag{4.16}$$

Equations (4.10), (4.14) and (4.16) prove the equivalence between the matrix model and the dual geometry.

Before ending this section, let us give an important remark. In [3] it was suggested that the total contribution to SO/Sp gauge theories should include both S^2 and RP^2 diagrams. Using this idea, explicit calculations have been carried out in [58] and it was found that at least up to order $\mathcal{O}(S^4)$, the whole result can be written as coming only from S^2 diagrams with modified color number. Later, a beautiful proof for SO group was given in in [64]. These observations are consistent with the result in the dual geometry where the integration of fluxes h around the origin is modified by the presence of the orientifold plane.

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